The Main Transcendental Question
First Part
How is pure mathematics possible?

§6

Here now is a great and proven body of cognition,[a] which is already of admirable extent and promises unlimited expansion in the future, which carries with it thoroughly apodictic certainty (i.e., absolute necessity), hence rests on no grounds of experience, and so is a pure product of reason, but beyond this is thoroughly synthetic. "How is it possible then for human reason to achieve such cognition wholly *a priori*?" Does not this capacity, since it is not, and cannot be, based on experience, presuppose some *a priori* basis for cognition, which lies deeply hidden, but which might reveal itself through these its effects, if their first beginnings were only diligently tracked down?

[4:281]

§7

We find, however, that all mathematical cognition has this distinguishing feature, that it must present its concept beforehand *in intuition* and indeed *a priori*, consequently in an intuition that is not empirical but pure, without which means it cannot take a single step; therefore its judgments are always *intuitive*,[b] in the place of which philosophy can content itself with *discursive*[c] judgments *from mere concepts*, and can indeed exemplify its

[a] *eine große und bewährte Erkenntnis* [b] *intuitiv* [c] *diskursiven*

apodictic teachings through intuition[d] but can never derive them from it. This observation with respect to the nature of mathematics already guides us toward the first and highest condition of its possibility; namely, it must be grounded in some *pure intuition* or other, in which it can present, or, as one calls it, *construct* all of its concepts *in concreto* yet *a priori*.* If we could discover this pure intuition and its possibility, then from there it could easily be explained how synthetic *a priori* propositions are possible in pure mathematics, and consequently also how this science itself is possible; for just as empirical intuition makes it possible for us, without difficulty, to amplify (synthetically in experience) the concept we form of an object of intuition through new predicates that are presented by intuition itself, so too will pure intuition do the same, only with this difference: that in the latter case the synthetic judgment will be *a priori* certain and apodictic, but in the former only *a posteriori* and empirically certain, because the former only contains what is met with in contingent empirical intuition, while the latter contains what necessarily must be met with in pure intuition, since it is, as intuition *a priori*, inseparably bound with the concept *before all experience* or individual perception.

§8

But with this step the difficulty seems to grow rather than diminish. For now the question runs: *How is it possible to intuit something* a priori? An intuition is a representation of the sort which would depend immediately on the presence of an object. It therefore seems impossible *originally* to [4:282] intuit *a priori*, since then the intuition would have to occur without an object being present, either previously or now, to which it could refer, and so it could not be an intuition. Concepts are indeed of the kind that we can quite well form some of them for ourselves *a priori* (namely, those that contain only the thinking of an object in general) without our being in an immediate relation to an object, e.g., the concept of magnitude, of cause, etc.; but even these still require, in order to provide them with signification and sense, a certain use *in concreto*, i.e., application to some intuition or other, by which an object for them is given to us. But how can the *intuition* of an object precede the object itself?

* See *Critique* p. 713.[1]

[d] *Anschauung*

[1] See pp. 195–6.

§9

If our intuition had to be of the kind that represented things *as they are in themselves*, then absolutely no intuition *a priori* would take place, but it would always be empirical. For I can only know what may be contained in the object in itself if the object is present and given to me. Of course, even then it is incomprehensible how the intuition of a thing that is present should allow me to cognize it the way it is in itself, since its properties cannot migrate over into my power of representation; but even granting such a possibility, the intuition still would not take place *a priori*, i.e., before the object were presented to me, for without that no basis for the relation of my representation to the object can be conceived; so it would have to be based on inspiration. There is therefore only one way possible for my intuition to precede the actuality of the object and occur as an *a priori* cognition, *namely if it contains nothing else except the form of sensibility, which in me as subject precedes all actual impressions through which I am affected by objects*. For I can know *a priori* that the objects of the senses can be intuited only in accordance with this form of sensibility. From this it follows: that propositions which relate merely to this form of sensory intuition will be possible and valid for objects of the senses; also, conversely, that intuitions which are possible *a priori* can never relate to things other than objects of our senses.

[4:283]

§10

Therefore it is only by means of the form of sensory intuition that we can intuit things *a priori*, though by this means we can cognize objects only as they *appear* to us (to our senses), not as they may be in themselves; and this supposition is utterly necessary, if synthetic propositions *a priori* are to be granted as possible, or, in case they are actually encountered, if their possibility is to be conceived and determined in advance.

Now space and time are the intuitions upon which pure mathematics bases all its cognitions and judgments, which come forward as at once apodictic and necessary; for mathematics must first exhibit all of its concepts in intuition – and pure mathematics in pure intuition – that is, it must first construct them, failing which (since mathematics cannot proceed analytically, namely, through the analysis of concepts, but only synthetically) it is impossible for it to advance a step, that is, as long as it lacks pure intuition, in which alone the material[e] for synthetic judgments *a priori* can be given. Geometry bases itself on the pure intuition of space. Even arithmetic forms its concepts of numbers through successive addition of units in time, but above all pure mechanics can form its concepts of motion only by means of the representation of time.[2] Both representations are, however, merely intuitions; for, if one eliminates from the empirical intuitions of bodies and their alterations (motion) everything empirical, that is, that which belongs to sensation, then space and time still remain, which are therefore pure intuitions that underlie *a priori* the empirical intuitions, and for that reason can never themselves be eliminated; but, by the very fact that they are pure intuitions *a priori*, they prove that they are mere forms of our sensibility that must precede all empirical intuition (i.e., the perception of actual objects), and in accordance with which objects can be cognized *a priori*, though of course only as they appear to us.

§11

The problem of the present section is therefore solved. Pure mathematics, as synthetic cognition *a priori*, is possible only because it refers to no other objects than mere objects of the senses, the empirical intuition of which is based on a pure and indeed *a priori* intuition (of space and time), and [4:284]
can be so based because this pure intuition is nothing but the mere form of sensibility, which precedes the actual appearance of objects, since it in fact first makes this appearance possible. This faculty of intuiting *a priori* does not, however, concern the matter of appearance – i.e., that which is sensation in the appearance, for that constitutes the empirical – but only the form of appearance, space and time. If anyone wishes to doubt in the slightest that the two are not determinations inhering in things in themselves but only mere determinations inhering in the relation of those things to sensibility, I would very much like to know how he can find it possible to know, *a priori* and therefore before all acquaintance with things, how their intuition must be constituted – which certainly is the case here with space and time. But this is completely comprehensible as soon as the two are taken for nothing more than formal conditions of

[e] *Stoff*
[2] Kant developed his analysis of motion and time in the *Metaphysical Foundations of Natural Science*.

our sensibility, and objects are taken merely for appearances; for then the form of appearance, i.e., the pure intuition, certainly can be represented from ourselves, i.e., *a priori*.

§12

In order to add something by way of illustration and confirmation, we need only to consider the usual and unavoidably necessary procedure of the geometers. All proofs of the thoroughgoing equality of two given figures (that one can in all parts be put in the place of the other) ultimately come down to this: that they are congruent with one another; which plainly is nothing other than a synthetic proposition based upon immediate intuition; and this intuition must be given pure and *a priori*, for otherwise that proposition could not be granted as apodictically certain but would have only empirical certainty. It would only mean: we observe it always to be so and the proposition holds only as far as our perception has reached until now. That full-standing space (a space that is itself not the boundary of another space)[3] has three dimensions, and that space in general cannot have more, is built upon the proposition that not more than three lines can cut each other at right angles in one point; this proposition can, however, by no means be proven from concepts, but rests immediately upon intuition, and indeed on pure *a priori* intuition,
[4:285] because it is apodictically certain; indeed, that we can require that a line should be drawn to infinity (*in indefinitum*), or that a series of alterations (e.g., spaces traversed through motion) should be continued to infinity, presupposes a representation of space and of time that can only inhere in intuition, that is, insofar as the latter is not in itself bounded by anything;[4] for this could never be concluded from concepts. Therefore pure intuitions *a priori* indeed actually do underlie mathematics, and make possible its synthetic and apodictically valid propositions; and consequently our transcendental deduction of the concepts of space and time[5] at the same time explains the possibility of a pure mathematics, a possibility which, without such a deduction, and without our assuming that "everything which our senses may be given (the outer in space, the inner in time) is

[3] In Euclid's *Elements* points are said to be the extremities or boundaries of lines and lines of planes (Bk. 1, defs. 3, 6, 13); planes are boundaries of spaces (Bk. 11, def. 2).

[4] See also Selections, pp. 159–60.

[5] For another mention of a deduction relating to space and time, see Selections, p. 168.

only intuited by us as it appears to us, not as it is in itself," could indeed be granted, but into which we could have no insight at all.

§13

All those who cannot yet get free of the conception, as if space and time were actual qualities attaching to things in themselves, can exercise their acuity on the following paradox, and, if they have sought its solution in vain, can then, free of prejudice at least for a few moments, suppose that perhaps the demotion of space and of time to mere forms of our sensory intuition may indeed have foundation.

incongruent counterparts

If two things are fully the same (in all determinations belonging to magnitude and quality) in all the parts of each that can always be cognized by itself alone, it should indeed then follow that one, in all cases and respects, can be put in the place of the other, without this exchange causing the least recognizable difference. In fact this is how things stand with plane figures in geometry; yet various spherical figures,[6] notwithstanding this sort of complete inner agreement, nonetheless reveal such a difference in outer relation that one cannot in any case be put in the place of the other; e.g., two spherical triangles from each of the hemispheres, which have an arc of the equator for a common base, can be fully equal with respect to their sides as well as their angles, so that nothing will be found in either, when it is fully described by itself, that is not also in the description of [4:286]
the other, and still one cannot be put in the place of the other (that is, in the opposite hemisphere); and here is then after all an *inner* difference between the triangles that no understanding can specify as inner, and that reveals itself only through the outer relation in space. But I will cite more familiar instances that can be taken from ordinary life.

What indeed can be more similar to, and in all parts more equal to, my hand or my ear than its image in the mirror? And yet I cannot put such a hand as is seen in the mirror in the place of its original; for if the one was a right hand, then the other in the mirror is a left, and the image of the right ear is a left one, which can never take the place of the former. Now there are no inner differences here that any understanding could merely think; and yet the differences are inner as far as the senses teach, for the left hand cannot, after all, be enclosed within the same boundaries as the

[6] A spherical figure is one inscribed in the surface of a sphere.

right (they cannot be made congruent), despite all reciprocal equality and similarity; one hand's glove cannot be used on the other. What then is the solution? These objects are surely not representations of things as they are in themselves, and as the pure understanding would cognize them, rather, they are sensory intuitions, i.e., appearances, whose possibility rests on the relation of certain things, unknown in themselves, to something else, namely our sensibility. Now, space is the form of outer intuition of this sensibility, and the inner determination of any space is possible only through the determination of the outer relation to the whole space of which the space is a part (the relation to outer sense); that is, the part is possible only through the whole, which never occurs with things in themselves as objects of the understanding alone, but does occur with mere appearances. We can therefore make the difference between similar and equal but nonetheless incongruent things (e.g., oppositely spiralled snails) intelligible through no concept alone, but only through the relation to right-hand and left-hand, which refers immediately to intuition.

[4:287]

Note I

Pure mathematics, and especially pure geometry, can have objective reality only under the single condition that it refers merely to objects of the senses, with regard to which objects, however, the principle remains fixed, that our sensory representation is by no means a representation of things in themselves, but only of the way in which they appear to us. From this it follows, not at all that the propositions of geometry are determinations of a mere figment of our poetic phantasy,[7] and therefore could not with certainty be referred to actual objects, but rather, that they are valid necessarily for space and consequently for everything that may be found in space, because space is nothing other than the form of all outer appearances, under which alone objects of the senses can be given to us. Sensibility, whose form lies at the foundation of geometry, is that upon which the possibility of outer appearances rests; these, therefore, can never contain anything other than what geometry prescribes to them. It would be completely different if the senses had to represent objects as they are in themselves. For then it absolutely would not follow from

[7] The word "phantasy" refers to the faculty of imagination.